Solutions to short-answer questions

1 a

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 12 & 8 \end{bmatrix}$$

b

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 8 & 8 \end{bmatrix}$$

2 Find the inverse of $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$.

$\text{Determinant} = 3 \times 8 - 4 \times 6 = 0$

This is a non-invertible matrix.

If $\mathbf{A} = \begin{bmatrix} x \\ y \end{bmatrix}$, then this corresponds to the simultaneous equations:

$3x + 4y = 8$
$6x + 8y = 16$

The second equation is equivalent to the first, as it is obtained by multiplying both sides of the first by 2.

Thus if $x = a$,

$$3a + 4y = 8$$
$$4y = 8 - 3a$$
$$y = 2 - \frac{3a}{4}$$

The matrices may be expressed as $\begin{bmatrix} a \\ 2 - \frac{3a}{4} \end{bmatrix}$.

3 a For a product to exist, the number of columns of the first matrix must equal the number of rows of the second. This is true only for $\mathbf{AC}$, $\mathbf{CD}$ and $\mathbf{BE}$, so these products exist.

b $$\mathbf{DA} = \begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \times 1 + 4 \times 3 \\ 2 \times 2 + 4 \times -1 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 0 \end{bmatrix}$$
$$\det(\mathbf{A}) = 1 \times -1 - 2 \times 3 = -7$$
$$\mathbf{A}^{-1} = \frac{1}{-7} \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{7} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$$

4 $$\mathbf{AB} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + -2 \times 1 + 1 \times 3 & 1 \times -4 + -2 \times -6 + 1 \times -8 \\ -5 \times 1 + 1 \times 1 + 2 \times 3 & -5 \times -4 + 1 \times -6 + 2 \times -8 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$
$$\det(\mathbf{C}) = 1 \times 4 - 2 \times 3 = -2$$
$$\mathbf{C}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

5 Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\text{Determinant} = 1 \times 4 - 2 \times 3 = -2$$
$$\text{Inverse} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Multiply by the inverse on the right:

$$\mathbf{A} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$$

6 $$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

7 The determinant must be zero.

$$\begin{aligned} 1 \times x - 2 \times 4 &= 0 \\ x - 8 &= 0 \\ x &= 8 \end{aligned}$$

8 a i $\mathbf{MM} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$

ii $\mathbf{MMM} = \mathbf{MM}(\mathbf{M})$

$= \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$

iii $\text{Determinant} = 2 \times 3 - -1 \times 1 = 7$

$\mathbf{M}^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

b $\mathbf{M}^{-1}\mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$= \frac{1}{7} \begin{bmatrix} 14 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x = 2,\ y = 1$

Solutions to multiple-choice questions

1 **B** The dimension is number of rows by number of columns, i.e. 4×2.

2 **E** The matrices cannot be added as they have different dimensions.

3 **C** $\mathbf{D} - \mathbf{C} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$

$- \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1-2 & -3 - -3 & 1-1 \\ 2-1 & 3-0 & -1 - -2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

4 **E** Multiply every entry by -1.

$-\mathbf{M} = - \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

5 **C** $2\mathbf{M} - 2\mathbf{N} = 2 \times \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 4 \\ -6 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 8 \\ 6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$

6 A $\mathbf{A}+\mathbf{B}$ will have the same dimension as $\mathbf{A}$ and $\mathbf{B}$, i.e. $m \times n$.

7 E The number of columns of $\mathbf{Q}$ is not the same as the number of rows of $\mathbf{P}$, so they cannot be multiplied.

8 A
$$\begin{aligned}\text{Determinant} &= 2 \times 1 - 2 \times -1 \\ &= 4\end{aligned}$$

9 E
$$\begin{aligned}\text{Determinant} &= 1 \times -2 - -1 \times 1 \\ &= -1 \\ \text{Inverse} &= \frac{1}{-1}\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\end{aligned}$$

10 D
$$\begin{aligned}\mathbf{NM} &= \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \times 0 + 2 \times -3 & 0 \times -2 + 2 \times 1 \\ 3 \times 0 + 1 \times -3 & 3 \times -2 + 1 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}\end{aligned}$$

Solutions to extended-response questions

1 a i The equations $2x - 3y = 3$ and $4x + y = 5$ can be written as

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

ii
$$\begin{aligned}\text{Determinant of } \mathbf{A} &= 2 \times 1 - 4 \times (-3) \\ &= 2 + 12 \\ &= 14 \\ \therefore \mathbf{A}^{-1} &= \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}\end{aligned}$$

iii
$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 18 \\ -2 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix}\end{aligned}$$

Therefore $x = \frac{9}{7}$ and $y = -\frac{1}{7}$.

iv The two lines corresponding to the equations intersect at $\left(\frac{9}{7}, -\frac{1}{7}\right)$.

b i The equations $2x + y = 3$ and $4x + 2y = 8$ can be written as

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

ii
$$\begin{aligned}\text{Determinant of } \mathbf{A} &= 2 \times 2 - 4 \times 1 \\ &= 4 - 4 \\ &= 0\end{aligned}$$

Since the determinant of $\mathbf{A}$ equals zero, $\mathbf{A}$ is a non-invertible matrix and the inverse $\mathbf{A}^{-1}$ does not exist.

c The two lines corresponding to the equations are parallel.

2 a The 2×3 matrix is: $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$

The rows correspond to the semesters and the columns to the forms of assessment.

b The percentages of the three components can be represented in the 3×1 matrix: $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Multiplying the two matrices gives the semester scores.

$$\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 79.2 \\ 80.4 \end{bmatrix}$$

Notice that multiplication of a 2×3 matrix by a 3×1 matrix results in a 2×1 matrix.

d For Chemistry the result is given by the following multiplication.

$$\begin{bmatrix} 86 & 82 & 84 \\ 81 & 80 & 70 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 83.8 \\ 75.2 \end{bmatrix}$$

e The aggregate of the four marks is 318.6. This is below 320.

f Three marks will be required to obtain an aggregate of marks above 320.

3 a The part-time and full-time teachers required for the 4 terms can be shown in a 4×2 matrix. The columns are for the two types of teachers and the rows for the different terms. Hence the matrix is: $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$

b The full-time teachers are paid $70 anhourandthepart - timeteachers$60. This can be represented in the 2×1 matrix: $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c

The product these two matrices gives the cost per hour for each term.

$$\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix}$$

The cost per hour for term 1 is $\$820$.

The cost per hour for term 2 is $\$800$.

The cost per hour for term 3 is $\$1040$.

The cost per hour for term 4 is $\$1020$.

d For the technical, catering and cleaning staff, the matrix for the 4 terms is the 4×3 matrix: $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$

e

The rate per hour can be represented in the 3×1 matrix: $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f The cost per hour is given by the product.

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix} = \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix}$$

The cost per hour for term 1 is $\$270$.

The cost per hour for term 2 is $\$270$.

The cost per hour for term 3 is $\$480$.

The cost per hour for term 4 is $\$480$.

g The total cost per hour is given by the sum of the matrices.

$$\begin{bmatrix} 820 \\ 800 \\ 1040 \\ 1020 \end{bmatrix} + \begin{bmatrix} 270 \\ 270 \\ 480 \\ 480 \end{bmatrix} = \begin{bmatrix} 1090 \\ 1070 \\ 1520 \\ 1500 \end{bmatrix}$$

The cost per hour for term 1 is $\$1090$.

The cost per hour for term 2 is $\$1070$.

The cost per hour for term 3 is $\$1520$.

The cost per hour for term 4 is $\$1500$.

4 **a** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

$\det(\mathbf{A}) = ad - bc$ and $\det(\mathbf{B}) = eh - fg$.

Then $\det(\mathbf{A})\det(\mathbf{B}) = (ad - bc)(eh - fg)$

$= adeh + bcfg - adfg - bceh$

Furthermore $\mathbf{AB} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.

and

$\det(\mathbf{AB}) = adeh + bcfg - adfg - bceh$

$\therefore \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

b A 2×2 matrix is invertible if and only if its determinant is non-zero. Hence if $\mathbf{A}$ and $\mathbf{B}$ are invertible then so is $\mathbf{AB}$